

POSET STRUCTURES IN BOIJ–SÖDERBERG THEORY

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ABSTRACT. Boij–Söderberg theory is the study of two cones: the cone of Betti diagrams of standard graded minimal free resolutions over a polynomial ring and the cone of cohomology tables of coherent sheaves over projective space. We provide a new interpretation of these partial orders in terms of the existence of nonzero homomorphisms, for both the general and equivariant constructions. These results provide new insights into the families of modules and sheaves at the heart of Boij–Söderberg theory: Cohen–Macaulay modules with pure resolutions and supernatural sheaves. In addition, they suggest the naturality of these partial orders and provide tools for extending Boij–Söderberg theory to other graded rings and projective varieties.

1. INTRODUCTION

Boij–Söderberg theory is the study of the cone of Betti diagrams over the standard graded polynomial ring $S = \mathbb{k}[x_1, \dots, x_n]$ and – dually – the cone of cohomology tables of coherent sheaves on $\mathbb{P}_{\mathbb{k}}^{n-1}$, where \mathbb{k} is a field. The extremal rays of these cones correspond to special modules and sheaves: Cohen–Macaulay modules with pure resolutions (Definition 2.1) and supernatural sheaves (Definition 5.1), respectively. Each set of extremal rays carries a partial order \preceq (Definitions 2.2 and 5.2) that induces a simplicial decomposition of the corresponding cone.

Each partial order \preceq is defined in terms of certain combinatorial data associated to these special modules and sheaves. For a module with a pure resolution, this data is a degree sequence, and for a supernatural sheaf, this data is a root sequence. Our main results reinterpret these partial orders \preceq in terms of the existence of nonzero homomorphisms between Cohen–Macaulay modules with pure resolutions and between supernatural sheaves.

Theorem 1.1. *Let ρ_d and $\rho_{d'}$ be extremal rays of the cone of Betti diagrams for S corresponding to Cohen–Macaulay modules with pure resolutions of types d and d' , respectively. Then $\rho_d \preceq \rho_{d'}$ if and only if there exist Cohen–Macaulay modules M and M' with pure resolutions of types d and d' , respectively, with $\mathrm{Hom}_S(M', M)_{\leq 0} \neq 0$.*

Theorem 1.2. *Let ρ_f and $\rho_{f'}$ be extremal rays of the cone of cohomology tables for \mathbb{P}^{n-1} corresponding to supernatural sheaves of types f and f' , respectively. Then $\rho_f \preceq \rho_{f'}$ if and only if there exist supernatural sheaves \mathcal{E} and \mathcal{E}' of types f and f' , respectively, with $\mathrm{Hom}_{\mathbb{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$.*

Though the statements of these two theorems are quite parallel, Theorem 1.1 is far more subtle than Theorem 1.2. Theorem 1.2 follows nearly directly from the Eisenbud–Schreyer pushforward construction of supernatural sheaves, but without modification, it is not clear how to compare the modules constructed in [ES09, §5].

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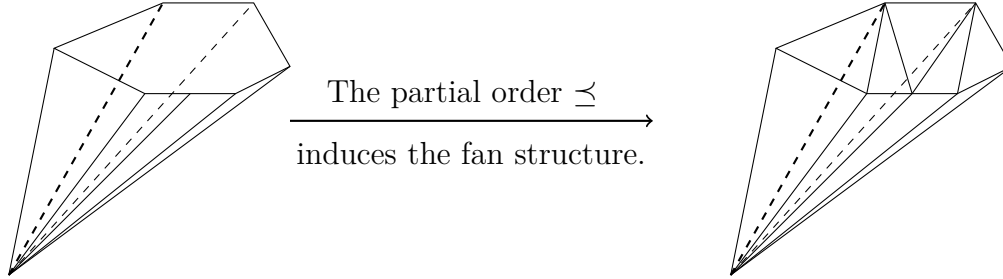


FIGURE 1. The partial order \preceq on the extremal rays induces a simplicial decomposition of the cone of Betti diagrams, where the simplices correspond to chains of extremal rays with respect to the partial order. This simplicial decomposition is essential to many applications of Boij–Söderberg theory.

We illustrate this via an example. Let $n = 3$, $d = (0, 2, 3, 5)$, $d' = (0, 3, 9, 10)$, and M and M' be finite length modules with pure resolutions of types d and d' , as constructed in [ES09, §5]. We know of no method to produce a nonzero element of $\text{Hom}(M, M')_{\leq 0}$, even in this specific case. The difficulty here stems from differences in the constructions of M and M' : the module M is constructed by pushing forward a complex of projective dimension 5 along $\mathbb{P}^2 \times (\mathbb{P}^1)^2 \rightarrow \mathbb{P}^2$, whereas M' is constructed by pushing forward a complex of projective dimension 10 along $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^5 \rightarrow \mathbb{P}^2$. Thus, the construction of [ES09, §5] does not even suggest that Theorem 1.1 ought to be true.

Our motivation for conjecturing the statement of Theorem 1.1 – and the first key idea behind its proof – is based on a flexible version of the Eisenbud–Schreyer construction of pure resolutions. This is Construction 3.3 below, and we show that the basic results of [ES09, §5] can be adapted to this construction. This extension enables us to use a single projection map to simultaneously produce modules N and N' with pure resolutions of types d and d' . In the case under consideration, we construct both N and N' by pushing forward complexes of projective dimension 10 along the projection map $\mathbb{P}^2 \times (\mathbb{P}^1)^7 \rightarrow \mathbb{P}^2$.¹

We may then produce elements of $\text{Hom}(N, N')_{\leq 0}$ by working with the complexes on the source $\mathbb{P}^2 \times (\mathbb{P}^1)^7$ of the projection map. However, finding such a nonzero element poses a second technical challenge in the proof of Theorem 1.1. This requires an explicit and somewhat delicate computation involving the pushforward of a morphism of complexes along the projection $\mathbb{P}^2 \times (\mathbb{P}^1)^7 \rightarrow \mathbb{P}^2$. This computation is carried out in the proof of Theorem 3.1, thus providing a new understanding of how certain modules with pure resolutions are related.

Besides providing greater insight into the structure of modules with pure resolutions and supernatural sheaves, our results have two further implications. First, the partial orders \preceq are defined in terms of the combinatorial data of degree sequences and root sequences (see Sections 2 and 5), and depend on the total order of \mathbb{Z} ; thus, they are only formally related to S and \mathbb{P}^{n-1} . However, our reinterpretations of \preceq in terms of module- and sheaf-theoretic properties suggest the naturality not only of \preceq , but also of the induced simplicial decompositions of both cones. In other words, while there exist graded modules whose Betti diagrams can be written as a positive sum of pure tables in several ways, Theorem 1.1 suggests that the most natural of these decompositions is the Boij–Söderberg decomposition produced by [ES09, Decomposition Algorithm], and similarly for Theorem 1.2 and cohomology tables.

¹We note that $M \neq N$ and $M' \neq N'$ in this example.

A second implication involves the extension of Boij–Söderberg theory to more complicated projective varieties or graded rings. For instance, the cone of free resolutions over a quadric hypersurface ring of $\mathbb{k}[x, y]$ is described in [BBEG11]. The extremal rays in this case correspond to pure resolutions of finite or infinite length. We could thus consider a partial order defined in parallel to Boij–Söderberg’s original definition (based on the combinatorial data of a degree sequence), or, following our result, we could consider a partial order defined in terms of nonzero homomorphisms. These partial orders are different in this hypersurface case; only the second definition leads to a decomposition algorithm for Betti diagrams. See Example 8.1 below for details.

For more general graded rings there even exist extremal rays that do not correspond to pure resolutions. (Similar statements hold for more general projective varieties.) There is thus no obvious extension of Boij–Söderberg’s original partial order to these cases. By contrast, the reinterpretations of \preceq provided by Theorems 1.1 and 1.2 are readily applicable to arbitrary projective varieties and graded rings. We discuss one such case in Example 8.2.

Theorems 1.1 and 1.2 hold over an arbitrary field \mathbb{k} , and their proofs involve variants of the constructions in [ES09] for supernatural sheaves and modules with pure resolutions. When $\text{char}(\mathbb{k}) = 0$, there also exist equivariant constructions of supernatural vector bundles [ES09, Thm. 6.2] and of finite length modules with pure resolutions [EFW11, Thm. 0.1]. For these we prove the most natural equivariant analogues of our main results.

Theorem 1.3. *Let V be an n -dimensional \mathbb{k} -vector space with $\text{char}(\mathbb{k}) = 0$, and let ρ_d and $\rho_{d'}$ be the extremal rays of the cone of Betti diagrams for $S = \text{Sym}(V)$ corresponding to finite length modules with pure resolutions of types d and d' . Then $\rho_d \preceq \rho_{d'}$ if and only if there exist finite length $\mathbf{GL}(V)$ -equivariant modules M and M' with pure resolutions of types d and d' , respectively, with $\text{Hom}_{\mathbf{GL}(V)}(M', M)_{\leq 0} \neq 0$.*

Theorem 1.4. *Let V be an n -dimensional \mathbb{k} -vector space with $\text{char}(\mathbb{k}) = 0$, and let ρ_f and $\rho_{f'}$ be the extremal rays of the cone of cohomology tables for $\mathbb{P}^{n-1} = \mathbb{P}(V)$ corresponding to supernatural vector bundles of types f and f' . Then $\rho_f \preceq \rho_{f'}$ if and only if there exist $\mathbf{GL}(V)$ -equivariant supernatural vector bundles \mathcal{E} and \mathcal{E}' of types f and f' , respectively, with $\text{Hom}_{\mathbf{GL}(V)}(\mathcal{E}', \mathcal{E}) \neq 0$.*

The action of $\mathbf{GL}(V)$ has two orbits on the maximal ideals of S : one consisting of the maximal ideal (x_1, \dots, x_n) and the other consisting of its complement. An equivariant Cohen–Macaulay module therefore has only two options for its support, and hence either has finite length or must be a free module. Thus the finite length hypothesis in Theorem 1.3 is the natural equivariant analogue of the Cohen–Macaulay hypothesis in Theorem 1.1.

As above, the statement for pure resolutions is more subtle than the corresponding statement for supernatural vector bundles. The modules constructed in [EFW11, §3] do not have nonzero equivariant homomorphisms between them, but the explicit combinatorics of the representation theory involved suggests a minor modification which does work. This also suggests how the maps should be defined in terms of the explicit presentation of the modules; the remaining nontrivial step is to show that these maps are in fact well-defined. The main obstacle is that such maps must be compatible with the actions of both the general linear group and the symmetric algebra, and the interplay between the two is delicate. This key issue in the proof of Theorem 1.3 is accomplished through a careful computation involving Pieri maps (combined with results from [SW11]).

Outline. In this paper, we first focus on the cone of Betti diagrams for S . In Section 2, we prove the reverse implications of Theorems 1.1 and 1.3. We then construct nonzero morphisms between modules with pure resolutions. Sections 3 and 4, respectively, address the forward directions of Theorems 1.1 and 1.3. We next address the cone of cohomology tables for \mathbb{P}^{n-1} . In Section 5, we prove the reverse implications of Theorems 1.2 and 1.4. We then turn to the construction of nonzero morphisms between supernatural sheaves: Sections 6 and 7, respectively, address the forward directions of Theorems 1.2 and 1.4. Finally, we provide in Section 8 a brief discussion of how Theorem 1.1 has been applied in the study of Boij–Söderberg theory over other graded rings. We suggest the survey [ES10b] to the reader seeking additional background on Boij–Söderberg theory.

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2. THE POSET OF DEGREE SEQUENCES

Let M be a finitely generated graded S -module. The (i, j) **th graded Betti number** of M , denoted $\beta_{i,j}(M)$, is $\dim_{\mathbb{k}} \operatorname{Tor}_i^S(\mathbb{k}, M)_j$. The **Betti diagram** of M is a table, with rows indexed by \mathbb{Z} and columns by $0, \dots, n$, such that the entry in column i and row j is $\beta_{i,i+j}(M)$. A sequence $d = (d_0, \dots, d_n) \in (\mathbb{Z} \cup \{\infty\})^{n+1}$ is called a **degree sequence** for S if $d_i > d_{i-1}$ for all i (with the convention that $\infty > \infty$). The **length** of d , denoted $\ell(d)$, is the largest integer t such that d_t is finite.

Definition 2.1. A graded S -module M is said to have a **pure resolution of type d** if a minimal free resolution of M has the form

$$0 \leftarrow M \leftarrow S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \dots \leftarrow S(-d_{\ell(d)})^{\beta_{\ell(d),d_{\ell(d)}}} \leftarrow 0. \quad \blacksquare$$

For every degree sequence d , there exists a Cohen–Macaulay module with a pure resolution of type d [ES09, Theorem 0.1] (see also [BS08a, Conjecture 2.4], [EFW11, Theorem 0.1]). The Betti diagram of any finitely generated S -module can be written as a positive rational combination of the Betti diagrams of Cohen–Macaulay modules with pure resolutions (see [ES09, Theorem 0.2] and [BS08b, Theorem 2]). The **cone of Betti diagrams** for S is the convex cone inside $\bigoplus_{j \in \mathbb{Z}} \mathbb{Q}^{n+1}$ generated by the Betti diagrams of all finitely generated S -modules. Each degree sequence d corresponds to a unique extremal ray of this cone, which we denote by ρ_d , and every extremal ray is of the form ρ_d for some degree sequence d .

Definition 2.2. For two degree sequences d and d' , we say that $d \preceq d'$ and that $\rho_d \preceq \rho_{d'}$ if $d_i \leq d'_i$ for all i . \blacksquare

This partial order induces a simplicial fan structure on the cone of Betti diagrams, where simplices correspond to chains of degree sequences under the partial order \preceq . We now show that the existence of a nonzero homomorphism between two modules with pure resolutions implies the comparability of their corresponding degree sequences. This result provides the reverse implications for Theorems 1.1 and 1.3.

Proposition 2.3. *Let M and M' be graded Cohen–Macaulay S -modules with pure resolutions of types d and d' , respectively. If $\text{Hom}(M', M)_{\leq 0} \neq 0$, then $d \preceq d'$.*

Proof. Write $\ell' = \ell(d')$ and $\ell = \ell(d)$. If $\ell' > \ell$, then $\text{codim } M' > \text{codim } M$, and, by [BH93, Propositions 1.2.3 and 1.2.1], $\text{Hom}(M', M) = 0$.

Therefore we may assume that $\ell' \leq \ell$. By hypothesis, we may fix a nonzero homomorphism $\phi \in \text{Hom}(M', M)_t$ for some $t \leq 0$. Let F_\bullet and F'_\bullet be minimal graded free resolutions of M and M' , respectively, and let $\{\phi_i: F'_i \rightarrow F_i\}_{i \geq 0}$ be the comparison maps in a lifting of ϕ . Suppose by way of contradiction that there is a j such that $d'_j < d_j$. Since $d'_j < d_j$, we see that $\phi_j = 0$. Hence, each ϕ_i such that $j \leq i \leq \ell'$ can be made zero by some homotopy equivalence. Write $(-)^{\vee} = \text{Hom}_S(-, S(-n))$. Since M and M' are Cohen–Macaulay, we note that $(F_\bullet)^{\vee}$ and $(F'_\bullet)^{\vee}$ are minimal graded free resolutions of $\text{Ext}_S^{\ell}(M, S(-n))$ and $\text{Ext}_S^{\ell'}(M', S(-n))$. Further, the maps $\{\phi_i^{\vee}\}_{i \geq 0}$ define an element of $\text{Ext}_S^{\ell-\ell'}(\text{Ext}_S^{\ell}(M, S(-n)), \text{Ext}_S^{\ell'}(M', S(-n)))$. In fact, if we write $N = \text{coker}((F'_{\ell'-1})^{\vee} \rightarrow (F_{\ell'}^{\vee})$, then $(\phi_{\ell'})^{\vee}: N \rightarrow \text{Ext}_S^{\ell'}(M', S(-n))$ is the zero homomorphism. Hence $\phi_i^{\vee} = 0$ for all $0 \leq i \leq \ell'$, and therefore $\phi = 0$. ■

Proposition 2.3 is untrue if we do not assume that M' is Cohen–Macaulay. For example, consider $S = \mathbb{k}[x, y]$, $M = S/\langle x^2 \rangle$, and $M' = S \oplus \mathbb{k}$. We used the hypothesis that M' is Cohen–Macaulay to have that $\text{codim } M' = \ell(d')$ and that $\text{Hom}_S(F'_\bullet, S(-n))$ is a resolution.

3. CONSTRUCTION OF MORPHISMS BETWEEN MODULES WITH PURE RESOLUTIONS

In Theorem 1.1 we must, necessarily, consider more than $\text{Hom}(M', M)_0$. For instance, if $n = 2$, $d = (0, 1, 2)$, and $d' = (1, 2, 3)$, then any M and M' with pure resolutions of types d and d' will be isomorphic to \mathbb{k}^m and $\mathbb{k}(-1)^{m'}$, respectively, for some integers m, m' . In this case, $\text{Hom}(M', M)_0 = 0$, whereas $\text{Hom}(M', M)_{-1} \neq 0$.

However, it is possible to reduce to the consideration of $\text{Hom}(M', M)_0$. To do this, let $t := \min\{d'_i - d_i \mid d'_i \neq \infty\}$. By replacing d' by $d' - (t, \dots, t)$, the forward direction of Theorem 1.1 is an immediate corollary of the following result.

Theorem 3.1. *Let $d \preceq d'$ be degree sequences for S with $d_j = d'_j$ for some $0 \leq j \leq \ell(d')$. Then there exist finitely generated graded Cohen–Macaulay modules M and M' with pure resolutions of types d and d' , respectively, with $\text{Hom}(M', M)_0 \neq 0$.*

Remark 3.2. The homomorphism group in Theorems 1.1 and 3.1 is nonzero only for specific choices of the modules M and M' . For two degree sequences $d \preceq d'$, there exist many pairs of modules M, M' with pure resolutions of types d and d' , respectively, such that $\text{Hom}(M', M)_{\leq 0} = 0$. For example, take $d = d' = (0, 2, 4)$, $M = S/\langle x^2, y^2 \rangle$, and $M' = S/\langle l_1^2, l_2^2 \rangle$ for general linear forms l_1 and l_2 . As another example, consider $d = (0, 3, 6) \prec d' = (0, 4, 8)$. When $M = S/\langle x^3, y^3 \rangle$ and $M' = S/\langle f, g \rangle$ for general quartic forms f and g , we again have $\text{Hom}(M', M)_{\leq 0} = 0$. ■

The proof of Theorem 3.1 is given at the end of this section and involves two main steps.

- (i) Construct twisted Koszul complexes \mathcal{K}_\bullet and \mathcal{K}'_\bullet on a product \mathbb{P} of projective spaces (including a copy of \mathbb{P}^{n-1}) and push them forward along the projection $\pi: \mathbb{P} \rightarrow \mathbb{P}^{n-1}$. This yields pure resolutions F_\bullet and F'_\bullet of types d and d' that respectively resolve modules M and M' .

- (ii) Show that there exists a morphism $h_\bullet: \mathcal{K}'_\bullet \rightarrow \mathcal{K}_\bullet$ such that the induced map $\nu_\bullet: F'_\bullet \rightarrow F_\bullet$ is not null-homotopic. This yields a nonzero element $\psi \in \text{Hom}_S(M', M)_0$.

We achieve (i) by modifying the construction of pure resolutions by Eisenbud and Schreyer [ES09, §5]. We replace their use of $\prod_i \mathbb{P}^{d_i - d_{i-1}}$ with a product of copies of \mathbb{P}^1 . This enables us to simultaneously construct pure resolutions of types d and d' and a nonzero map between the modules they resolve. The details of (i) are contained in Construction 3.3. For (ii), we apply Construction 3.3 so as to produce the morphism h_\bullet . Checking that the induced map ν_\bullet is not null-homotopic uses, in an essential way, the hypothesis that $d_j = d'_j$ for some $0 \leq j \leq \ell(d')$. Example 3.5 demonstrates these arguments. Write $\mathbb{P}^{1 \times r}$ for the r -fold product of \mathbb{P}^1 .

Construction 3.3 (Modification of the Eisenbud–Schreyer construction of pure resolutions). The objects involved in this construction of a pure resolution F_\bullet of type d will be denoted by Kos_\bullet^d , \mathcal{K}_\bullet , and \mathcal{L} . The corresponding objects for the pure resolution F'_\bullet of type d' are $\text{Kos}_\bullet^{d'}$, \mathcal{K}'_\bullet , and \mathcal{L}' . Let

$$(3.4) \quad r := \max\{d_{\ell(d)} - d_0 - \ell(d), d'_{\ell(d')} - d_0 - \ell(d')\}$$

and $\mathbb{P} := \mathbb{P}^{n-1} \times \mathbb{P}^{1 \times r}$. On \mathbb{P} , fix the coordinates

$$([x_1 : x_2 : \cdots : x_n], [y_0^{(1)} : y_1^{(1)}], \dots, [y_0^{(r)} : y_1^{(r)}])$$

and consider the multilinear forms

$$f_p := \sum_{i_0 + \cdots + i_r = p} \left(x_{i_0} \cdot \prod_{j=1}^r y_{i_j}^{(j)} \right) \quad \text{for } p = 1, 2, \dots, n + r.$$

(Note that $i_0 \in \{1, \dots, n\}$ and $i_j \in \{0, 1\}$ for all $1 \leq j \leq r$.) We now define

$$\begin{aligned} D &:= \{d_0, d_0 + 1, \dots, d_0 + \ell(d) + r\}, & D' &:= \{d_0, d_0 + 1, \dots, d_0 + \ell(d') + r\}, \\ \delta &:= (\delta_1 < \cdots < \delta_r) = D \setminus d, & \delta' &:= (\delta'_1 < \cdots < \delta'_r) = D' \setminus d', \\ a &:= \delta - (d_0 + 1, \dots, d_0 + 1), & a' &:= \delta' - (d_0 + 1, \dots, d_0 + 1), \\ \mathcal{L} &:= \mathcal{O}_{\mathbb{P}}(-d_0, a), & \text{and } \mathcal{L}' &:= \mathcal{O}_{\mathbb{P}}(-d_0, a'). \end{aligned}$$

(We view δ and δ' as ordered sequences.) Let Kos_\bullet^d be the Koszul complex on $f_1, \dots, f_{\ell(d)+r}$, which is an acyclic complex of sheaves on \mathbb{P} of length $\ell(d) + r$ (see [ES09, Proposition 5.2]). Let $\mathcal{K}_\bullet := \text{Kos}_\bullet^d \otimes \mathcal{L}$. Let $\pi: \mathbb{P} \rightarrow \mathbb{P}^{n-1}$ denote the projection onto the first factor. By repeated application of [ES09, Proposition 5.3], $\pi_* \mathcal{K}_\bullet$ is an acyclic complex of sheaves on \mathbb{P}^{n-1} of length $\ell(d)$ such that each term is a direct sum of line bundles. Taking global sections of this complex in all twists yields the pure resolution F_\bullet of a graded S -module (that is finitely generated and Cohen–Macaulay). We can write the free module F_i explicitly as follows. If $s = \max\{i \mid a_i - d_j + d_0 \leq -2\}$, then we have

$$F_j = S(-d_j)^{\binom{\ell(d)+r}{d_j - d_0}} \otimes \left(\bigotimes_{i=1}^s H^1(\mathbb{P}^1, \mathcal{O}(a_i - d_j + d_0)) \right) \otimes \left(\bigotimes_{i=s+1}^r H^0(\mathbb{P}^1, \mathcal{O}(a_i - d_j + d_0)) \right).$$

Let $\text{Kos}_\bullet^{d'}$ be the Koszul complex on $f_1, \dots, f_{\ell(d')+r}$ and $\mathcal{K}'_\bullet := \text{Kos}_\bullet^{d'} \otimes \mathcal{L}'$, and define F'_\bullet in a similar manner. ■

The value of r in (3.4) is the least integer such that we are able to fit both the twists $-d_0$ and $\min\{-d_{\ell(d)}, -d'_{\ell(d')}\}$ in the \mathbb{P}^{n-1} coordinate of the bundles of the complexes \mathcal{K}_\bullet and \mathcal{K}'_\bullet . The choices of a and a' , which ensure that F_\bullet and F'_\bullet are pure of types d and d' , are dictated by the homological degrees in \mathcal{K}_\bullet and \mathcal{K}'_\bullet that need to be eliminated in each projection away from a \mathbb{P}^1 component of \mathbb{P} . In Example 3.5, these homological degrees are those with an underlined -1 in Table 1. Observe that $a - a' \in \mathbb{N}^r$ since $d \preceq d'$. Thus there is a nonzero map $h_\bullet: \mathcal{K}'_\bullet \rightarrow \mathcal{K}_\bullet$ that is induced by a polynomial of multidegree $(0, a - a')$. In (ii), we show that $\pi_* h_\bullet$ induces the desired nonzero map.

The following extended example contains all of the main ideas behind the proof of Theorem 3.1.

Example 3.5. Consider $d = (0, 2, 4, 5, 6)$ and $d' = (1, 2, 4, 7) = (1, 2, 4, 7, \infty)$. Note that $d_2 = d'_2 = 4$, so that d and d' satisfy the hypotheses of Theorem 3.1. Here $r = 4$ and $\mathbb{P} = \mathbb{P}^3 \times \mathbb{P}^{1 \times 4}$. On \mathbb{P} , we have the Koszul complexes $\text{Kos}_\bullet^d = \text{Kos}_\bullet(\mathcal{O}_\mathbb{P}; f_1, \dots, f_8)$ and $\text{Kos}_\bullet^{d'} = \text{Kos}_\bullet(\mathcal{O}_\mathbb{P}; f_1, \dots, f_7)$. There is a natural map $\text{Kos}_\bullet^{d'} \rightarrow \text{Kos}_\bullet^d$ induced by the inclusion $\langle f_1, \dots, f_7 \rangle \subseteq \langle f_1, \dots, f_8 \rangle$. Here we have

$$\begin{aligned} \delta &= (1, 3, 7, 8), & \delta' &= (0, 3, 5, 6), \\ a &= (0, 2, 6, 7), & a' &= (-1, 2, 4, 5), \\ \mathcal{K}_\bullet &= \text{Kos}_\bullet^d \otimes \mathcal{O}_\mathbb{P}(0, a), & \text{and } \mathcal{K}'_\bullet &= \text{Kos}_\bullet^{d'} \otimes \mathcal{O}_\mathbb{P}(0, a'). \end{aligned}$$

Table 1 shows the twists in each homological degree of these complexes.

$d = (0, 2, 4, 5, 6)$		$d' = (1, 2, 4, 7)$	
i	Twist in \mathcal{K}_i	i	Twist in \mathcal{K}'_i
0	$(0, 0, 2, 6, 7)$	0	$(0, \underline{-1}, 2, 4, 5)$
-1	$(-1, \underline{-1}, 1, 5, 6)$	-1	$(-1, -2, 1, 3, 4)$
-2	$(-2, -2, 0, 4, 5)$	-2	$(-2, -3, 0, 2, 3)$
-3	$(-3, -3, \underline{-1}, 3, 4)$	-3	$(-3, -4, \underline{-1}, 1, 2)$
-4	$(-4, -4, -2, 2, 3)$	-4	$(-4, -5, -2, 0, 1)$
-5	$(-5, -5, -3, 1, 2)$	-5	$(-5, -6, -3, \underline{-1}, 0)$
-6	$(-6, -6, -4, 0, 1)$	-6	$(-6, -7, -4, -2, \underline{-1})$
-7	$(-7, -7, -5, \underline{-1}, 0)$	-7	$(-7, -8, -5, -3, -2)$
-8	$(-8, -8, -6, -2, \underline{-1})$		

TABLE 1. Twists appearing in \mathcal{K}_\bullet and \mathcal{K}'_\bullet in Example 3.5.

Let h be a nonzero homogeneous polynomial on \mathbb{P} of multidegree $(0, a - a') = (0, 1, 0, 2, 2)$. Then multiplication by h induces a nonzero map $h: \mathcal{K}'_0 \rightarrow \mathcal{K}_0$. To write h , we use matrix multi-index notation for the monomials in $\mathbb{k}[y_0^{(1)}, y_1^{(1)}, \dots, y_0^{(4)}, y_1^{(4)}]$, where the i th column represents the multi-index of the $y^{(i)}$ -coordinates. With this convention, fix

$$h = \mathbf{y}^{\begin{pmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} := y_0^{(1)} \cdot (y_0^{(3)})^2 \cdot (y_0^{(4)})^2.$$

Denote the induced map of complexes $\mathcal{K}'_\bullet \rightarrow \mathcal{K}_\bullet$ by h_\bullet . Taking the direct image of h_\bullet along the natural projection $\pi: \mathbb{P} \rightarrow \mathbb{P}^3$ and its global sections in all twists induces a map $\nu_\bullet: F'_\bullet \rightarrow F_\bullet$.

We claim that ν_\bullet is not null-homotopic. This need not hold for an arbitrary pair $d \preceq d'$, however it does hold for a pair of degree sequences which satisfy the hypotheses of Theorem 3.1. We use the fact that $d_2 = d'_2 = 4$, as this implies that $\nu_2: F'_2 \rightarrow F_2$ is a matrix of scalars. Since F'_\bullet and F_\bullet are both minimal free resolutions, it then follows that the map ν_2 factors through a null-homotopy only if ν_2 is itself the zero map. Thus it is enough to show that $\nu_2 \neq 0$. For this, note that

$$F_2 = S(-4)^{\binom{8}{4}} \otimes H^1(\mathbb{P}^1, \mathcal{O}(-4)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(2)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(3))$$

and $F'_2 = S(-4)^{\binom{7}{4}} \otimes H^1(\mathbb{P}^1, \mathcal{O}(-5)) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(0)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(1))$

and that F_2 and F'_2 have H^1 terms in precisely the same positions, and similarly for the H^0 terms. We may then use [BEKS11, Lemma 7.3] to compute the map $\nu_2: F'_2 \rightarrow F_2$ explicitly. Since the matrix is too large to be written down, we simply exhibit a basis element of F'_2 that is not mapped to zero.

For $I = \{i_1 < \dots < i_4\}$ a subset of either $\{1, \dots, 8\}$ or $\{1, \dots, 7\}$, we use the notation $\epsilon_I := \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_4}$ to write S -bases for $S(-4)^{\binom{8}{4}}$ and for $S(-4)^{\binom{7}{4}}$. Choose the natural monomial bases for the cohomology groups appearing in the tensor product expressions for F_2 and F'_2 , and write these monomials in multi-index notation. Recalling the above definition of h , we then have that

$$\epsilon_{1,2,3,4} \otimes \mathbf{y}^{\binom{-4 \ -1 \ 0 \ 1}{-1 \ -1 \ 0 \ 0}}$$

is a basis element of F_2 . We compute

$$\begin{aligned} \nu_2 \left(\epsilon_{1,2,3,4} \otimes \mathbf{y}^{\binom{-4 \ -1 \ 0 \ 1}{-1 \ -1 \ 0 \ 0}} \right) &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\binom{-4 \ -1 \ 0 \ 1}{-1 \ -1 \ 0 \ 0}} \cdot h \\ &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\binom{-4 \ -1 \ 0 \ 1}{-1 \ -1 \ 0 \ 0} + \binom{1 \ 0 \ 2 \ 2}{0 \ 0 \ 0 \ 0}} \\ &= \epsilon_{1,2,3,4} \otimes \mathbf{y}^{\binom{-3 \ -1 \ 2 \ 3}{-1 \ -1 \ 0 \ 0}}. \end{aligned}$$

Since this yields a basis element of F'_2 , it is clear that ν_2 is a nonzero map, so ν_\bullet is not null-homotopic. \blacksquare

Proof of Theorem 3.1. Construction 3.3 yields finitely generated graded Cohen–Macaulay modules M and M' that have pure resolutions F_\bullet and F'_\bullet of types d and d' , respectively. To construct the desired nonzero map $\psi: M' \rightarrow M$, we fix a generic homogeneous form h on \mathbb{P} of multidegree $(0, a - a')$, which exists because $a - a' = \delta - \delta' \in \mathbb{N}^r$. Multiplication by h induces a map $h_\bullet: \mathcal{K}'_\bullet \rightarrow \mathcal{K}_\bullet$. The functoriality of π_* induces a map $\pi_* \mathcal{K}'_\bullet \rightarrow \pi_* \mathcal{K}_\bullet$ that, upon taking global sections in all twists, yields a map $\nu_\bullet: F'_\bullet \rightarrow F_\bullet$. Let $\psi: M' \rightarrow M$ be the map induced by ν_\bullet .

To show that ψ is nonzero, it suffices to show that ν_\bullet is not null-homotopic. Let j be the index such that $d_j = d'_j$. Then F_j and F'_j are generated entirely in the same degree. Since F_\bullet and F'_\bullet are minimal free resolutions, $\nu_j: F'_j \rightarrow F_j$ is given by a matrix of scalars. Thus it follows that ν_\bullet is null-homotopic only if ν_j is the zero map. We now use the description of ν_j given in [BEKS11, Lemma 7.3]. (The relevant homological degree in both \mathcal{K}_\bullet and \mathcal{K}'_\bullet is $d_j - d_0$.)

Let $s = \max\{i \mid a_i - d_j + d_0 \leq -2\}$ and let $s' = \max\{i \mid a'_i - d'_j + d_0 \leq -2\}$. Note that, since $d_j = d'_j$, the construction of a and a' implies that $s = s'$. We then have

$$F_j = S(-d_j)^{\binom{\ell(d)+r}{d_j-d_0}} \otimes \left(\bigotimes_{i=1}^s H^1(\mathbb{P}^1, \mathcal{O}(a_i - d_j + d_0)) \right) \otimes \left(\bigotimes_{i=s+1}^r H^0(\mathbb{P}^1, \mathcal{O}(a_i - d_j + d_0)) \right) \text{ and}$$

$$F'_j = S(-d_j)^{\binom{\ell(d')+r}{d_j-d_0}} \otimes \left(\bigotimes_{i=1}^s H^1(\mathbb{P}^1, \mathcal{O}(a'_i - d_j + d_0)) \right) \otimes \left(\bigotimes_{i=s+1}^r H^0(\mathbb{P}^1, \mathcal{O}(a'_i - d_j + d_0)) \right),$$

where both F_j and F'_j have the same number of factors involving H^0 (and therefore also the same number involving H^1). Hence we can repeatedly apply [BEKS11, Lemma 7.3] to conclude that ν_j is simply the map induced on cohomology by the map $h_{d_j-d_0}: \mathcal{K}'_{d_j-d_0} \rightarrow \mathcal{K}_{d_j-d_0}$.

We now fix a specific value of h and show that $\nu_j \neq 0$. Let $c := a - a' \in \mathbb{N}^r$ and write $c = (c_1, \dots, c_r)$. Let

$$h := (y_0^{(1)})^{c_1} \cdot (y_0^{(2)})^{c_2} \cdots (y_0^{(r)})^{c_r} = \mathbf{y}^{\binom{c_1 \dots c_r}{0 \dots 0}},$$

so that h is the unique monomial of multidegree $(0, c)$ that involves only the $y_0^{(i)}$ -variables.

For $I = \{i_1 < \dots < i_{d_j-d_0}\}$ a subset of either $\{1, \dots, \ell(d) + r\}$ or $\{1, \dots, \ell(d') + r\}$, we use the notation $\epsilon_I := \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_{d_j-d_0}}$ to write S -bases for $S(-d_j)^{\binom{\ell(d)+r}{d_j-d_0}}$ and for $S(-d_j)^{\binom{\ell(d')+r}{d_j-d_0}}$. Choose the natural monomial bases for the cohomology groups appearing in the tensor product expression for F_j and F'_j , and write these monomials in matrix multi-index notation, as in Example 3.5. For each i corresponding to an H^1 -term (i.e. $i \in \{1, \dots, s\}$), let $u_i := -(a_i - d_j + d_0) + 1$. For each i corresponding to an H^0 term (i.e. $i \in \{s+1, \dots, r\}$), let $w_i := -(a_i - d_j + d_0)$. Observe that

$$\epsilon_{\{1, \dots, d_j-d_0\}} \otimes \mathbf{y}^{\binom{u_1 \dots u_s \ w_{s+1} \dots w_r}{-1 \dots -1 \ 0 \dots 0}}$$

is a basis element of F_j . We then have that

$$\begin{aligned} \nu_j \left(\epsilon_{\{1, \dots, d_j-d_0\}} \otimes \mathbf{y}^{\binom{u_1 \dots u_s \ w_{s+1} \dots w_r}{-1 \dots -1 \ 0 \dots 0}} \right) &= \epsilon_{\{1, \dots, d_j-d_0\}} \otimes \mathbf{y}^{\binom{u_1 \dots u_s \ w_{s+1} \dots w_r}{-1 \dots -1 \ 0 \dots 0}} \cdot h \\ &= \epsilon_{\{1, \dots, d_j-d_0\}} \otimes \mathbf{y}^{\binom{u_1 \dots u_s \ w_{s+1} \dots w_r}{-1 \dots -1 \ 0 \dots 0}} \cdot \mathbf{y}^{\binom{c_1 \dots c_r}{0 \dots 0}} \\ &= \epsilon_{\{1, \dots, d_j-d_0\}} \otimes \mathbf{y}^{\binom{u_1+c_1 \dots u_s+c_s \ w_{s+1}+c_{s+1} \dots w_r+c_r}{-1 \dots -1 \ 0 \dots 0}}. \end{aligned}$$

One may check that this is a basis element of F'_j , and hence the map ν_j is nonzero. Therefore ν_\bullet is not null-homotopic, as desired. \blacksquare

4. EQUIVARIANT CONSTRUCTION OF MORPHISMS BETWEEN MODULES WITH PURE RESOLUTIONS

Throughout this section, we assume that \mathbb{k} is a field of characteristic 0 and that all degree sequences have length n . Let V be an n -dimensional \mathbb{k} -vector space, and let $S = \text{Sym}(V)$. We use \mathbf{S}_λ to denote a Schur functor, as in Section 7. As in Section 3, a shift of d' reduces the remaining direction of Theorem 1.3 to the following result.

Theorem 4.1. *Let $d \preceq d'$ be two degree sequences such that $d_k = d'_k$ for some k . Then there exist finite length $\mathbf{GL}(V)$ -equivariant S -modules M and M' with pure resolutions of types d and d' , respectively, with $\mathrm{Hom}_{\mathbf{GL}(V)}(M', M)_0 \neq 0$.*

Our proof of Theorem 4.1 relies on Lemma 4.2, which handles the special case when the degree sequences d and d' differ by 1 in a single position. This proof will repeatedly appeal to Pieri's rule for decomposing the tensor product of a Schur functor by a symmetric power. We refer the reader to [SW11, §1.1 and Theorem 1.3] for a statement of this rule, as our main use of it will be through [SW11, Lemma 1.6].

Given a degree sequence d , let $M(d)$ be the $\mathbf{GL}(V)$ -equivariant graded S -module constructed in [EFW11, §3] (see also [SW11, §2.1]), and let $\mathbf{F}(d)_\bullet$ be its $\mathbf{GL}(V)$ -equivariant free resolution. By construction, the generators for each S -module $\mathbf{F}(d)_j$ form an irreducible $\mathbf{GL}(V)$ -module whose highest weight we call $\lambda(d)_j$. For instance, if $d = (0, 2, 5, 7, 8)$, then $\lambda(d)_0 = (3, 1, 0, 0)$ and $\lambda(d)_1 = (5, 1, 0, 0)$ [EFW11, Example 3.3]. Note that $M(d) \otimes V$ is also an equivariant module with a pure resolution of type d .

Lemma 4.2. *Let $d = (d_0, \dots, d_n) \in \mathbb{Z}^{n+1}$ be a degree sequence, and let d' be the degree sequence obtained from d by replacing d_i by $d_i + 1$ for some i . Then there exists an equivariant nonzero morphism $\phi: M(d') \otimes V \rightarrow M(d)$.*

Further, if F_\bullet and F'_\bullet are the minimal free resolutions of $M(d)$ and $M(d') \otimes V$ respectively, then we may choose ϕ so that the induced map $F'_j \rightarrow F_j$ is surjective for all $j \neq i$.

Remark 4.3. Let d and d' be degree sequences as in the statement of Lemma 4.2. We observe that

- (i) $\lambda(d')_i = \lambda(d)_i$.
 - (ii) If $j < i$, then $\lambda(d')_j$ is obtained from $\lambda(d)_j$ by removing a box from the i th part.
 - (iii) If $j > i$, then $\lambda(d')_j$ is obtained from $\lambda(d)_j$ by removing a box from the $(i+1)$ st part.
- For instance, if $d = (0, 2, 4)$ and $d' = (0, 3, 4)$, then we have

$$\lambda(d)_j = \begin{cases} (1, 0) & \text{if } j = 0 \\ (3, 0) & \text{if } j = 1 \\ (3, 2) & \text{if } j = 2 \end{cases} \quad \text{and} \quad \lambda(d')_j = \begin{cases} (0, 0) & \text{if } j = 0 \\ (3, 0) & \text{if } j = 1 \\ (3, 1) & \text{if } j = 2. \end{cases} \quad \blacksquare$$

Remark 4.4. In the proof of Lemma 4.2, we repeatedly use [SW11, Lemma 1.6]. The statement of the lemma is for factorizations of Pieri maps into simple Pieri maps $\mathbf{S}_\nu V \rightarrow \mathbf{S}_\eta V \otimes V$, but we need to factor into simple Pieri maps as well as simple co-Pieri maps $\mathbf{S}_\eta V \otimes V \rightarrow \mathbf{S}_\nu V$. No modification of the proof is needed: we simply use the fact that the composition of a co-Pieri map and a Pieri map of the same type is an isomorphism and that in each case that we apply [SW11, Lemma 1.6], the Pieri maps may be factored so that the simple Pieri maps and simple co-Pieri maps of the same type appear consecutively. \blacksquare

Proof of Lemma 4.2. Set $\lambda_\ell = \sum_{j=\ell}^{n-1} (d_{j+1} - d_j - 1)$ for $1 \leq \ell \leq n-1$, $\lambda_n = 0$, $\mu_1 = \lambda_1 + d_1 - d_0$, and $\mu_\ell = \lambda_\ell$ for $1 \leq \ell \leq n$. If $i = n$, we modify λ and μ by adding 1 to all of its parts (so in particular, $\lambda_n = \mu_n = 1$). As in [EFW11, §3], define M to be the cokernel of the Pieri map

$$\psi_{\mu/\lambda}: S(-d_1) \otimes \mathbf{S}_\mu V \rightarrow S(-d_0) \otimes \mathbf{S}_\lambda V.$$

We will choose partitions λ' and μ' so that M' is the cokernel of the Pieri map

$$\psi_{\mu'/\lambda'}: S(-d'_1) \otimes \mathbf{S}_{\mu'} V \rightarrow S(-d'_0) \otimes \mathbf{S}_{\lambda'} V.$$

To do this, we separately consider the three cases $i = 0$, $i = 1$, and $i \geq 2$. In each case, we specify λ' and μ' (these descriptions are special cases of Remark 4.3) and construct a commutative diagram of equivariant degree 0 maps

$$(4.5) \quad \begin{array}{ccc} S(-d_1) \otimes \mathbf{S}_\mu V & \xrightarrow{\psi_{\mu/\lambda}} & S(-d_0) \otimes \mathbf{S}_\lambda V \\ \phi_\mu \uparrow & & \uparrow \phi_\lambda \\ S(-d'_1) \otimes \mathbf{S}_{\mu'} V \otimes V & \xrightarrow{\psi_{\mu'/\lambda'} \otimes 1_V} & S(-d'_0) \otimes \mathbf{S}_{\lambda'} V \otimes V \end{array}$$

that induces an equivariant degree 0 map of the cokernels $\phi: M' \rightarrow M$. Since the Pieri maps are only well-defined up to a choice of nonzero scalar, we only prove that the square commutes up to a choice of nonzero scalar. One may scale appropriately to obtain strict commutativity.

Finally, after handling the three separate cases, we prove that the induced maps $F'_j \rightarrow F_j$ are surjective whenever $j \neq i$. Since F'_\bullet is a minimal free resolution, this implies that the map $F'_\bullet \rightarrow F_\bullet$ is not null-homotopic, and hence $\phi: M' \rightarrow M$ is nonzero.

Case $i = 1$. Set $\lambda'_1 = \lambda_1 - 1$, $\lambda'_j = \lambda_j$ for $2 \leq j \leq n$, and $\mu' = \mu$. Also, let $d'_0 = d_0$ and $d'_1 = d_1 + 1$. Using the notation of (4.5), we define ϕ_μ by identifying $\mathbf{S}_{\mu'} V \otimes V$ with $\text{Sym}^1 V \otimes \mathbf{S}_\mu V$ and then extending it to an S -linear map. Let ϕ_λ be the projection of $\mathbf{S}_{\lambda'} V \otimes V \rightarrow \mathbf{S}_\lambda V$ tensored with the identity of $S(-d_0)$. From the degree $d_1 + 1$ part of (4.5), we obtain

$$\begin{array}{ccc} \text{Sym}^1 V \otimes \mathbf{S}_\mu V & \xrightarrow{\alpha} & \text{Sym}^{d_1-d_0+1} V \otimes \mathbf{S}_\lambda V \\ \beta \uparrow & & \uparrow \gamma \\ \mathbf{S}_\mu V \otimes V & \xrightarrow{\delta} & \text{Sym}^{d_1-d_0+1} V \otimes \mathbf{S}_{\lambda'} V \otimes V. \end{array}$$

Note that α is the linear part of $\mathbf{F}_1 \rightarrow \mathbf{F}_0$ and is hence injective because $d_2 - d_1 > 1$. Since β is an isomorphism, $\alpha\beta$ is injective. Also we have $\lambda_1 > \lambda_2$ because $d_2 - d_1 > 1$, so by Pieri's rule, every summand of $\mathbf{S}_\mu V \otimes V$ is also a summand of $\text{Sym}^{d_1-d_0+1} V \otimes \mathbf{S}_\lambda V$. Using [SW11, Lemma 1.6], one can show that $\gamma\delta$ is also injective. Since the tensor product $\text{Sym}^{d_1-d_0+1} V \otimes \mathbf{S}_\lambda V$ is multiplicity-free by the Pieri rule, this implies that these maps are equal after rescaling the image of each direct summand of $\mathbf{S}_\mu V \otimes V$ by some nonzero scalar. Hence this diagram is commutative, and the same is true for (4.5).

Case $i \geq 2$. Set $\lambda'_i = \lambda_i - 1$ and $\lambda_j = \lambda_j$ for $j \neq i$. Similarly, set $\mu'_i = \mu_i - 1$ and $\mu'_j = \mu_j$ for $j \neq i$. Using the notation of (4.5), let ϕ_μ be a nonzero projection of $\mathbf{S}_{\mu'} V \otimes V$ onto $\mathbf{S}_\mu V$ tensored with the identity on $S(-d_1)$. Similar to the previous case, choose a nonzero projection $\mathbf{S}_{\lambda'} V \otimes V \rightarrow \mathbf{S}_\lambda V$ and tensor it with the identity map on $S(-d_0)$ to get ϕ_λ . From the degree d_1 part of (4.5), we obtain

$$\begin{array}{ccc} \mathbf{S}_\mu V & \xrightarrow{\alpha} & \text{Sym}^{d_1-d_0} V \otimes \mathbf{S}_\lambda V \\ \beta \uparrow & & \uparrow \gamma \\ \mathbf{S}_{\mu'} V \otimes V & \xrightarrow{\delta} & \text{Sym}^{d_1-d_0} V \otimes \mathbf{S}_{\lambda'} V \otimes V. \end{array}$$

Let $\mathbf{S}_\nu V$ be a direct summand of $\mathbf{S}_{\mu'} V \otimes V$. If $\nu \neq \mu$, then $\mathbf{S}_\nu V$ is not a summand of $\text{Sym}^{d_1-d_0} V \otimes \mathbf{S}_\lambda V$, as otherwise we would have $\nu_i = \lambda_i - 1$, and both of the compositions $\alpha\beta$ and $\gamma\delta$ would therefore be 0 on such a summand. If $\nu = \mu$, then the composition $\alpha\beta$ is nonzero, so it is enough to check that the same is true for $\gamma\delta$; this holds by [SW11, Lemma 1.6], and hence this diagram and (4.5) are commutative.

Case $i = 0$. Set $d^\vee := (-d_n, -d_{n-1}, \dots, -d_0)$ and $d'^\vee := (-d'_n, -d'_{n-1}, \dots, -d'_0)$. Since $d_j = d'_j$ for all $j \neq i = 0$, we see that d^\vee and d'^\vee only differ in position n . Hence, by the case $i \geq 2$ above (we assume that $n \geq 2$ since the $n = 1$ case is easily done directly), we have finite length modules $M(d^\vee)$ and $M(d'^\vee)$ with pure resolutions of types d^\vee and d'^\vee , respectively, along with a nonzero morphism $\psi: M(d^\vee) \otimes V \rightarrow M(d'^\vee)$. If we define $N^\vee := \text{Ext}^n(N, S)$, then $M(d'^\vee)^\vee \cong M(d')$ and $(M(d^\vee) \otimes V)^\vee \cong M(d) \otimes V^*$ (both isomorphisms are up to some power of $\wedge^n V$ which we cancel off). In addition, since $\text{Ext}^n(-, S)$ is a duality functor on the space of finite length S -modules, we obtain a nonzero map

$$\psi^\vee: M(d') \rightarrow M(d) \otimes V^*.$$

By adjunction, we then obtain a nonzero map $M(d') \otimes V \rightarrow M(d)$.

Fixing some $j \neq i$, we now prove the surjectivity of the maps $F'_j \rightarrow F_j$, which implies that ϕ is a nonzero morphism, as observed above. The key observation is that, in each of the above three cases, F_j is an irreducible Schur module. Since $d_j = d'_j$, the map

$$F'_j = S(-d'_j) \otimes \mathbf{S}_{\lambda(d'_j)} V \otimes V \rightarrow F_j = S(-d_j) \otimes \mathbf{S}_{\lambda(d_j)} V$$

is induced by a nonzero equivariant map $\mathbf{S}_{\lambda(d'_j)} V \otimes V \rightarrow \mathbf{S}_{\lambda(d_j)} V$. Since the target is an irreducible representation, this morphism, and hence the map $F'_j \rightarrow F_j$, is surjective. More specifically, the map $\mathbf{S}_{\lambda(d'_j)} V \otimes V \rightarrow \mathbf{S}_{\lambda(d_j)} V$ is a projection onto one of the factors in the Pieri rule decomposition of $\mathbf{S}_{\lambda(d'_j)} V \otimes V$. \blacksquare

Example 4.6. This example illustrates the construction of Lemma 4.2 when $d = (0, 2, 4)$ and $d' = (0, 3, 4)$. When writing the free resolutions, we simply write the Young diagram of λ in place of the corresponding graded equivariant free module. Also, we follow the conventions in [EFW11] and [SW11] and draw the Young diagram of λ by placing λ_i boxes in the i th column, rather than the usual convention of using rows. The morphism from Lemma 4.2 yields a map of complexes, which we write as

$$\begin{array}{ccccccc} M & \longleftarrow & \square & \longleftarrow & \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} & \longleftarrow & \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \longleftarrow 0 \\ \uparrow \psi & & \uparrow & & \uparrow & & \uparrow \\ M' & \longleftarrow & \square \otimes \emptyset & \longleftarrow & \square \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} & \longleftarrow & \square \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \longleftarrow 0. \end{array}$$

Observe that $d_2 = 4 = d'_2$ and that the vertical arrow in homological position 2 is surjective, as it corresponds to a Pieri rule projection. A similar statement holds in position 0. \blacksquare

Proof of Theorem 4.1. Set $r := \sum_{j=0}^n d'_j - d_j$. We may construct a sequence of degree sequences $d =: d^0 < d^1 < \dots < d^r := d'$ such that d^j and d^{j+1} satisfy the hypotheses of

Lemma 4.2 for any j . Lemma 4.2 yields a nonzero morphism

$$\phi^{(j+1)}: M(d^{j+1}) \otimes V \rightarrow M(d^j)$$

for any $j = 1, \dots, r$. If we set $M^{(j)} := M(d^j) \otimes V^{\otimes j}$, and we set $\psi^{(j+1)}$ to be the natural map

$$\psi^{(j+1)}: M^{(j+1)} \rightarrow M^{(j)}$$

given by $\phi^{(j)} \otimes \text{id}_V^{\otimes j}$, then we may compose the map $\psi^{(j+1)}$ with the map $\psi^{(j)}$.

Let $M := M^{(0)} = M(d)$, and let $M' := M^{(r)} = M(d') \otimes V^{\otimes r}$. We then have an equivariant map $\psi := \psi^{(1)} \circ \dots \circ \psi^{(r)}: M' \rightarrow M$, and we must finally show that ψ is nonzero. Let $F_{\bullet}^{(j)}$ be the minimal free resolution of $M^{(j)}$. Since $d_k = d'_k$, it follows that $d_k^{(j)} = d_k^{(j+1)}$ for all j . Lemma 4.2 then implies that we can choose each $\phi^{(j+1)}$ such that the map $\psi^{(j+1)}$ induces a surjection $F_k^{(j+1)} \rightarrow F_k^{(j)}$. Since the composition of surjective maps is surjective, it follows that the map $F_k^{(r)} \rightarrow F_k^{(0)}$ induced by ψ is surjective. Since $F_{\bullet}^{(0)}$ is a minimal free resolution, we conclude that the map of complexes $F_{\bullet}^{(r)} \rightarrow F_{\bullet}^{(0)}$ is not null-homotopic, and hence $\psi: M' \rightarrow M$ is a nonzero morphism. ■

Remark 4.7. By introducing a variant of Lemma 4.2, we may simplify the construction used in the proof of Theorem 4.1. Let d and d' be two degree sequences such that $d'_i = d_i + N$, and $d'_j = d_j$ for all $j \neq i$. Iteratively applying Lemma 4.2 yields a morphism $\phi: M(d') \otimes V^{\otimes N} \rightarrow M(d)$. Since $\text{char}(\mathbb{k}) = 0$, we have an inclusion $\iota: \text{Sym}^N V \rightarrow V^{\otimes N}$, and we let ψ be the morphism induced by composing ϕ and $\text{id}_{M(d')} \otimes \iota$. Let F'_{\bullet} and F_{\bullet} be the minimal free resolutions of $M(d') \otimes \text{Sym}^N V$ and $M(d)$ respectively. The map $F'_j \rightarrow F_j$ induced by ψ is induced by the equivariant map of vector spaces

$$\mathbf{S}_{\lambda(d')} V \otimes \text{Sym}^N V \rightarrow \mathbf{S}_{\lambda(d)} V.$$

This map is surjective because it is a projection onto one of the factors in the Pieri rule decomposition of $\mathbf{S}_{\lambda(d')} V \otimes \text{Sym}^N V$.

This simplifies the proof of Theorem 4.1 as follows. Let $i_1 > \dots > i_{\ell}$ be the indices for which d and d' differ. By iteratively applying the construction outlined in this remark, we may construct the desired modules and nonzero morphism in ℓ steps. Since ℓ can be far smaller than $r := \sum_{j=0}^n d'_j - d_j$, this variant is useful for computing examples such as Example 4.8. ■

Example 4.8. We illustrate Theorem 4.1 with $n = 4$, $d = (0, 2, 3, 6, 7)$, and $d' = (1, 2, 5, 6, 10)$. Using the notation of Remark 4.7, $d^{(1)} = (0, 2, 3, 6, 10)$, $d^{(2)} = (0, 2, 5, 6, 10)$. Following the same conventions as in Example 4.6, the corresponding resolutions are given in Figure 2. Notice that $d_3 = 6 = d'_3$. Focusing on the third terms of the resolutions, we see that the maps are simply projections from Pieri's rule. In particular, these maps are surjective and therefore nonzero. ■

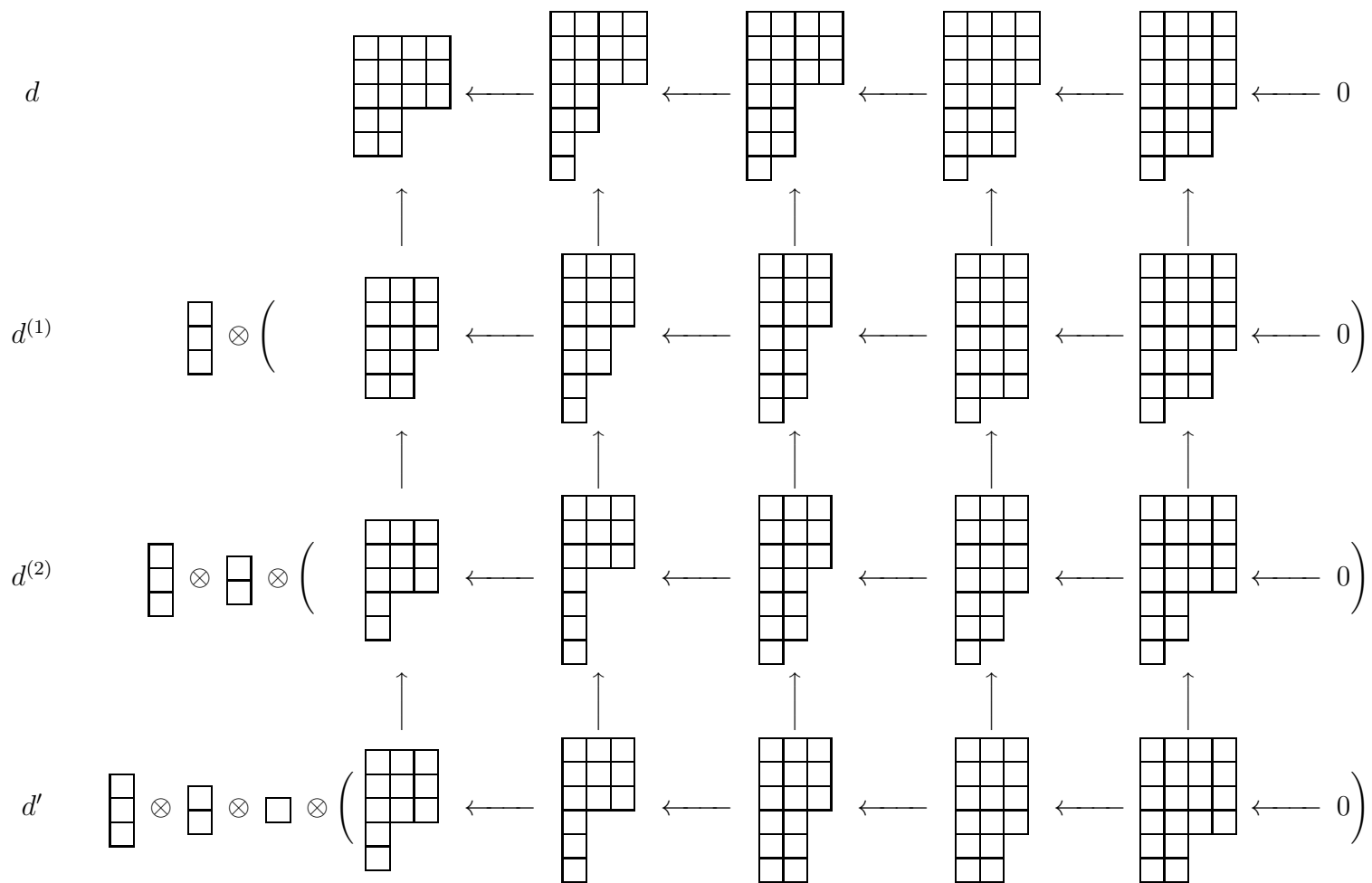


FIGURE 2. The Young diagram depictions of the resolutions in Example 4.8.

5. THE POSET OF ROOT SEQUENCES

Let \mathcal{E} be a coherent sheaf on \mathbb{P}^{n-1} . The **cohomology table** of \mathcal{E} is a table with rows indexed by $\{0, \dots, n-1\}$ and columns indexed by \mathbb{Z} , such that the entry in row i and column j is $\dim_{\mathbb{k}} H^i(\mathbb{P}^{n-1}, \mathcal{E}(j-i))$. A sequence $f = (f_1, \dots, f_{n-1}) \in (\mathbb{Z} \cup \{-\infty\})^{n-1}$ is called a **root sequence** for \mathbb{P}^{n-1} if $f_i < f_{i-1}$ for all i (with the convention that $-\infty < -\infty$). The **length** of f , denoted $\ell(f)$, is the largest integer t such that f_t is finite.

Definition 5.1. Let f be a root sequence for \mathbb{P}^{n-1} . A sheaf \mathcal{E} on \mathbb{P}^{n-1} is **supernatural of type $f = (f_1, \dots, f_{n-1})$** if the following are satisfied:

- (i) The dimension of $\text{Supp } \mathcal{E}$ is $\ell(f)$.
- (ii) For all $j \in \mathbb{Z}$, there exists at most one i such that $\dim_{\mathbb{k}} H^i(\mathbb{P}^{n-1}, \mathcal{E}(j)) \neq 0$.
- (iii) The Hilbert polynomial of \mathcal{E} has roots $f_1, \dots, f_{\ell(f)}$.

Dropping the reference to its root sequence, we also say that \mathcal{E} is a **supernatural sheaf** (or a **supernatural vector bundle** if it is locally free). ■

For every root sequence f , there exists a supernatural sheaf of type f [ES09, Theorem 0.4]. Moreover, the cohomology table of any coherent sheaf can be written as a positive real combination of cohomology tables of supernatural sheaves [ES10a, Theorem 0.1]. The **cone of cohomology tables** for \mathbb{P}^{n-1} is the convex cone inside $\prod_{j \in \mathbb{Z}} \mathbb{R}^n$ generated by cohomology tables of coherent sheaves on \mathbb{P}^{n-1} . Each root sequence f corresponds to a unique extremal ray of this cone, which we denote by ρ_f , and every extremal ray is of the form ρ_f for some root sequence f .

Definition 5.2. For two root sequences f and f' , we say that $f \preceq f'$ and that $\rho_f \preceq \rho_{f'}$ if $f_i \leq f'_i$ for all i . ■

This partial order induces a simplicial fan structure on the cone of cohomology tables, where simplices correspond to chains of root sequences under the partial order \preceq . We now show that the existence of a nonzero homomorphism between two supernatural sheaves implies the comparability of their corresponding root sequences, which provides the reverse implications for Theorems 1.2 and 1.4.

Proposition 5.3. *Let \mathcal{E} and \mathcal{E}' be supernatural sheaves of types f and f' respectively. If $\text{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$, then $f \preceq f'$.*

Proof. Let $T(\mathcal{E})$ and $T(\mathcal{E}')$ denote the Tate resolutions of \mathcal{E} and \mathcal{E}' [EFS03, §4]. These are doubly infinite acyclic complexes over the exterior algebra Λ , which is Koszul dual to S and has generators in degree -1 . Since $\text{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$, there is a map $\phi: T(\mathcal{E}') \rightarrow T(\mathcal{E})$ that is not null-homotopic. Observe that for every cohomological degree j , $\phi^j: T(\mathcal{E}')^j \rightarrow T(\mathcal{E})^j$ is nonzero. First, if $\phi^j = 0$ for some j , then, we may take $\phi^k = 0$ for all $k < j$. Secondly, if $k > j$, then after applying $\text{Hom}_{\Lambda}(-, \Lambda)$ (which is exact because Λ is self-injective), we can take ϕ^k to be zero.

By [ES09, Theorem 6.4], we see that all the minimal generators of $T(\mathcal{E})^j$ (respectively, $T(\mathcal{E}')^j$) are of a single degree i (respectively, i'). (This is equivalent to stating that every column of the cohomology table of \mathcal{E} and \mathcal{E}' contains precisely one nonzero entry.) Since ϕ^j is nonzero and Λ is generated in elements of degree -1 , we see that $i' \leq i$. Now, again by [ES09, Theorem 6.4], $f \preceq f'$. ■

6. CONSTRUCTION OF MORPHISMS BETWEEN SUPERNATURAL SHEAVES

The goal of this section is to prove Theorem 6.1, which provides the forward direction of Theorem 1.2.

Theorem 6.1. *Let $f \preceq f'$ be two root sequences. Then there exist supernatural sheaves \mathcal{E} and \mathcal{E}' of types f and f' , respectively, with $\text{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.*

For the purposes of exposition, we separate the proof of Theorem 6.1 into two cases (with $\ell(f) = \ell(f')$ and with $\ell(f) < \ell(f')$), and handle these cases in Propositions 6.8 and 6.12 respectively. Examples 6.4 and 6.9 illustrate the essential ideas behind the proof in each case.

If $\ell(f) < n - 1$, then we call $(f_1, \dots, f_{\ell(f)})$ the **truncation** of f , and write $\tau(f)$. Let $f = (f_1, \dots, f_{n-1})$ be a root sequence with $\ell(f) = s$. Denote the s -fold product of \mathbb{P}^1 by $\mathbb{P}^{1 \times s}$. Fix homogeneous coordinates

$$(6.2) \quad ([y_0^{(1)} : y_1^{(1)}], \dots, [y_0^{(s)} : y_1^{(s)}]) \quad \text{on} \quad \mathbb{P}^{1 \times s}.$$

In order to produce a supernatural sheaf of type f on \mathbb{P}^{n-1} , we first construct a supernatural vector bundle of type $\tau(f)$ on \mathbb{P}^s . Its image under an embedding of \mathbb{P}^s as a linear subvariety \mathbb{P}^{n-1} will give the desired supernatural sheaf.

We now outline our approach to construct a nonzero map between supernatural sheaves on \mathbb{P}^s of types $f \preceq f'$ in the case that $\ell(f) = \ell(f') = s$. This uses the proof of [ES09, Theorem 6.1].

- (i) Construct a finite map $\pi: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^s$.
- (ii) Choose appropriate line bundles \mathcal{L} and \mathcal{L}' on $\mathbb{P}^{1 \times s}$ so that $\pi_* \mathcal{L}$ and $\pi_* \mathcal{L}'$ are supernatural vector bundles of the desired types.
- (iii) When $\ell(f) = \ell(f') = s$, construct a morphism $\mathcal{L}' \xrightarrow{\phi} \mathcal{L}$ such that $\pi_* \phi$ is nonzero.

For (i), we use the multilinear $(1, \dots, 1)$ -forms

$$(6.3) \quad g_p := \sum_{i_1 + \dots + i_s = p} \left(\prod_{j=1}^s y_{i_j}^{(j)} \right) \quad \text{for} \quad p = 0, \dots, s$$

on $\mathbb{P}^{1 \times s}$ to define the map $\pi: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^s$ via $[g_0 : \dots : g_s]$. For (ii), with $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^s$,

$$\mathcal{E}_f := \pi_* (\mathcal{O}_{\mathbb{P}^{1 \times s}}(-f - \mathbf{1}))$$

is a supernatural vector bundle of type $\tau(f)$ on \mathbb{P}^s of rank $s!$ (the degree of π). The next example illustrates (iii).

Example 6.4. Here we find a nonzero morphism $\mathcal{E}_{f'} \rightarrow \mathcal{E}_f$ that is the direct image of a morphism of line bundles on $\mathbb{P}^{1 \times (n-1)}$. Let $n = 5$ and $f := (-2, -3, -4, -5) \preceq f' := (-1, -2, -3, -4)$. The map $\pi: \mathbb{P}^{1 \times 4} \rightarrow \mathbb{P}^4$ is finite of degree $4! = 24$. Following steps (i) and (ii) as outlined above, we set $\mathcal{E} := \mathcal{E}_f = \pi_* \mathcal{O}_{\mathbb{P}^{1 \times 4}}(1, 2, 3, 4)$ and $\mathcal{E}' := \mathcal{E}_{f'} = \pi_* \mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3)$. There is a natural inclusion

$$(6.5) \quad \pi_* \text{Hom}_{\mathbb{P}^{1 \times 4}} (\mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3), \mathcal{O}_{\mathbb{P}^{1 \times 4}}(1, 2, 3, 4)) \subseteq \text{Hom}_{\mathbb{P}^4} (\mathcal{E}', \mathcal{E}),$$

which induces an inclusion of global sections (see Remark 6.6). Therefore

$$\begin{aligned} \mathrm{Hom}(\mathcal{E}', \mathcal{E}) &\supseteq H^0(\mathbb{P}^4, \pi_* \mathcal{H}om_{\mathbb{P}^{1 \times 4}}(\mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3), \mathcal{O}_{\mathbb{P}^{1 \times 4}}(1, 2, 3, 4))) \\ &= H^0(\mathbb{P}^{1 \times 4}, \mathcal{O}_{\mathbb{P}^{1 \times 4}}(1, 1, 1, 1)) \\ &\simeq \mathbb{K}^{16}. \end{aligned}$$

We thus conclude that $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.

The inclusion (6.5) is strict. Note that, by definition, neither \mathcal{E}' nor \mathcal{E} has intermediate cohomology, and hence, by Horrocks' Splitting Criterion, both \mathcal{E} and \mathcal{E}' must split as the sum of line bundles. Thus $\mathcal{E}' = \mathcal{O}_{\mathbb{P}^4}^{24}$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^4}(1)^{24}$, and it follows that $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) = H^0(\mathbb{P}^4, \mathcal{O}(1)^{576}) \simeq \mathbb{K}^{2880}$. ■

Remark 6.6. Let $\pi: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^s$ be as in (i). For coherent sheaves \mathcal{F} and \mathcal{G} on $\mathbb{P}^{1 \times s}$, we have

$$\pi_* \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^{1 \times s}}}(\mathcal{F}, \mathcal{G}) \subseteq \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^s}}(\pi_* \mathcal{F}, \pi_* \mathcal{G}).$$

Indeed, this can be checked locally. Let $U \subseteq \mathbb{P}^s$ be an affine open subset, and write $A = H^0(U, \mathcal{O}_{\mathbb{P}^s})$ and $B = H^0(U, \pi_* \mathcal{O}_{\mathbb{P}^{1 \times s}})$. For all B -modules M and N , every nonzero B -module homomorphism is also a nonzero A -module homomorphism via the map $A \rightarrow B$. Injectivity is immediate. ■

Remark 6.7. Suppose that $\beta: \mathbb{P}^s \rightarrow \mathbb{P}^{n-1}$ is a closed immersion as a linear subvariety. Let \mathcal{E} be a coherent sheaf on \mathbb{P}^s . It follows from the projection formula and from the finiteness of β that \mathcal{E} is a supernatural sheaf on \mathbb{P}^s of type (f_1, \dots, f_s) if and only if $\beta_* \mathcal{E}$ is a supernatural sheaf on \mathbb{P}^{n-1} of type $(f_1, \dots, f_s, -\infty, \dots, -\infty)$. ■

Proposition 6.8. *If $\ell(f) = \ell(f')$, then Theorem 6.1 holds.*

Proof. We first reduce to the case $\ell(f') = n - 1$. Let $\beta: \mathbb{P}^{\ell(f')} \rightarrow \mathbb{P}^{n-1}$ be a closed immersion as a linear subvariety. Let $\ell(f') = s$ and write $f = (f_1, \dots, f_s, -\infty, \dots, -\infty)$ and $f' = (f'_1, \dots, f'_s, -\infty, \dots, -\infty)$. Assume that \mathcal{E} and \mathcal{E}' are supernatural sheaves of type (f_1, \dots, f_s) and (f'_1, \dots, f'_s) on \mathbb{P}^s and that $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$. Then, by Remark 6.7, $\beta_* \mathcal{E}$ and $\beta_* \mathcal{E}'$ are supernatural sheaves of types f and f' , and $\mathrm{Hom}(\beta_* \mathcal{E}', \beta_* \mathcal{E}) \neq 0$.

We may thus assume that $\ell(f') = n - 1$. Let $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^{n-1}$. Let $\pi: \mathbb{P}^{1 \times (n-1)} \rightarrow \mathbb{P}^{n-1}$ be the morphism given by the forms g_p defined in (6.3) (with $s = n - 1$). Let $\mathcal{E} := \mathcal{E}_f = \pi_* \mathcal{O}(-f - \mathbf{1})$ and $\mathcal{E}' := \mathcal{E}_{f'} = \pi_* \mathcal{O}(-f' - \mathbf{1})$. Remark 6.6 shows that

$$H^0(\mathbb{P}^{n-1}, \pi_* \mathcal{H}om_{\mathbb{P}^{1 \times (n-1)}}(\mathcal{O}(-f' - \mathbf{1}), \mathcal{O}(-f - \mathbf{1}))) \subseteq \mathrm{Hom}_{\mathbb{P}^{n-1}}(\mathcal{E}', \mathcal{E}).$$

Note that $\mathcal{H}om_{\mathbb{P}^{1 \times (n-1)}}(\mathcal{O}(-f' - \mathbf{1}), \mathcal{O}(-f - \mathbf{1})) = \mathcal{O}(f' - f)$. Since $f \preceq f'$, we have that $H^0(\mathbb{P}^{1 \times (n-1)}, \mathcal{O}(f' - f)) \neq 0$, and thus $\mathrm{Hom}_{\mathbb{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$. ■

When $\ell(f) < \ell(f')$, the supernatural sheaves constructed using (i) and (ii) above have supports of different dimensions. Before addressing this general case, we provide an example.

Example 6.9. Let $n = 5$ and $f = (-2, -3, -4, -\infty) \preceq f' = (-1, -2, -3, -4)$, so that $\ell(f) = 3 < \ell(f') = 4 = n - 1$. We proceed by modifying steps (i)-(iii) above.

(i') We extend the construction of (i) to the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^{1 \times 3} & \xrightarrow{\alpha} & \mathbb{P}^{1 \times 4} \\ \downarrow \pi^{(3)} & & \downarrow \pi^{(4)} \\ \mathbb{P}^3 & \xrightarrow{\beta} & \mathbb{P}^4. \end{array}$$

(ii') Choose appropriate line bundles \mathcal{L} on $\mathbb{P}^{1 \times 3}$ and \mathcal{L}' on $\mathbb{P}^{1 \times 4}$, so that $\pi_*^{(3)} \mathcal{L}$ and $\pi_*^{(4)} \mathcal{L}'$ are supernatural sheaves of the desired types.

(iii') Construct a morphism $\mathcal{L}' \xrightarrow{\phi} \alpha_* \mathcal{L}$ such that $\pi_*^{(4)} \phi$ is nonzero.

For (i'), we use the homogeneous coordinates from (6.2). The maps $\pi^{(3)}$ and $\pi^{(4)}$ are instances of the map π from (i) for $\mathbb{P}^{1 \times 3}$ and $\mathbb{P}^{1 \times 4}$, respectively. Define a closed immersion $\alpha: \mathbb{P}^{1 \times 3} \rightarrow \mathbb{P}^{1 \times 4}$ by the vanishing of the coordinate $y_1^{(4)}$. Fix coordinates x_0, \dots, x_4 for \mathbb{P}^4 , and let $\beta: \mathbb{P}^3 \rightarrow \mathbb{P}^4$ be the closed immersion given by the vanishing of x_4 . We now have that the diagram in (i') is indeed commutative.

In (ii'), we take $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 2, 3)$ and $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3)$ and set $\mathcal{E}_f = \pi_*^{(3)} \mathcal{L}$ and $\mathcal{E}_{f'} = \pi_*^{(4)} \mathcal{L}'$. Set $\mathcal{E} := \beta_* \mathcal{E}_f$ and $\mathcal{E}' := \mathcal{E}_{f'}$. Then \mathcal{E} is a supernatural sheaf on \mathbb{P}^4 (see Remark 6.7), and

$$\mathrm{Hom}_{\mathbb{P}^4}(\mathcal{E}', \mathcal{E}) = H^0\left(\mathbb{P}^4, \mathrm{Hom}\left(\pi_*^{(4)}(\mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3)), \pi_*^{(4)}(\alpha_* \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 2, 3))\right)\right).$$

By Remarks 6.6 and 6.10, we obtain the containment

$$\begin{aligned} \mathrm{Hom}_{\mathbb{P}^4}(\mathcal{E}', \mathcal{E}) &\supseteq H^0\left(\mathbb{P}^4, \pi_*^{(4)} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3), \alpha_* \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 2, 3))\right) \\ &\cong H^0\left(\mathbb{P}^{1 \times 4}, \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^{1 \times 4}}(0, 1, 2, 3), \alpha_* \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 2, 3))\right) \\ &\cong H^0\left(\mathbb{P}^{1 \times 4}, (\alpha_* \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 1, 1))(0, 0, 0, -3)\right) \\ &\cong H^0\left(\mathbb{P}^{1 \times 4}, \alpha_* \mathcal{O}_{\mathbb{P}^{1 \times 3}}(1, 1, 1)\right) \cong \mathbb{k}^8. \end{aligned}$$

In particular, $\mathrm{Hom}_{\mathbb{P}^4}(\mathcal{E}', \mathcal{E}) \neq 0$, as desired. ■

Remark 6.10. Let $1 \leq s < t$, and let $\alpha: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^{1 \times t}$ be the embedding given by the vanishing of $y_1^{(s+1)}, \dots, y_1^{(t)}$. Let \mathcal{F} be a coherent sheaf on $\mathbb{P}^{1 \times s}$ and $b \in \mathbb{Z}^{t-s}$. Write $\mathbf{0}_s$ for the 0-vector in \mathbb{Z}^s . Then

$$(6.11) \quad H^i\left(\mathbb{P}^{1 \times t}, (\alpha_* \mathcal{F})(\mathbf{0}_s, b)\right) \cong H^i\left(\mathbb{P}^{1 \times t}, \alpha_* \mathcal{F}\right) \cong H^i\left(\mathbb{P}^{1 \times s}, \mathcal{F}\right)$$

The first isomorphism follows from the projection formula, taken along with the fact that, by the definition of α , the line bundle $\mathcal{O}_{\mathbb{P}^{1 \times t}}(\mathbf{0}_s, b)$ is trivial when restricted to the support of $\alpha_* \mathcal{F}$ (which is contained in $\mathbb{P}^{1 \times s}$). The second isomorphism holds because α is a finite morphism. ■

Proposition 6.12. *If $\ell(f) < \ell(f')$, then Theorem 6.1 holds.*

Proof. We may reduce to the case $\ell(f') = n - 1$ by the same argument as in the beginning of the proof of Proposition 6.8.

Let $s = \ell(f)$ and consider the line bundles $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{1 \times s}}(-\tau(f) - \mathbf{1})$ on $\mathbb{P}^{1 \times s}$ and $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^{1 \times (n-1)}}(-f' - \mathbf{1})$ on $\mathbb{P}^{1 \times (n-1)}$. Let $\pi: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^s$ and $\pi': \mathbb{P}^{1 \times (n-1)} \rightarrow \mathbb{P}^{n-1}$ be the maps defined by the forms in (6.3). Let $\mathcal{E}_f = \pi_* \mathcal{L}$ and $\mathcal{E}_{f'} = (\pi')_* \mathcal{L}'$, and define the closed

immersion $\alpha: \mathbb{P}^{1 \times s} \rightarrow \mathbb{P}^{1 \times (n-1)}$ by the vanishing of the coordinates $y_1^{(s+1)}, \dots, y_1^{(n-1)}$. Fix coordinates x_0, \dots, x_{n-1} for \mathbb{P}^{n-1} , and let $\beta: \mathbb{P}^s \rightarrow \mathbb{P}^{n-1}$ be the closed immersion given by the vanishing of x_{s+1}, \dots, x_{n-1} . This yields the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^{1 \times s} & \xrightarrow{\alpha} & \mathbb{P}^{1 \times (n-1)} \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{P}^s & \xrightarrow{\beta} & \mathbb{P}^{n-1}. \end{array}$$

By Remark 6.7, $\mathcal{E} := \beta_* \mathcal{E}_f$ is a supernatural sheaf of type f . Also, $\mathcal{E}' := \mathcal{E}_{f'}$ is a supernatural sheaf of type f' .

We must show that $\mathrm{Hom}_{\mathbb{P}^{n-1}}(\mathcal{E}', \mathcal{E}) \neq 0$. It suffices to show that $\mathrm{Hom}_{\mathbb{P}^{1 \times (n-1)}}(\mathcal{L}', \alpha_* \mathcal{L}) \neq 0$ by Remark 6.6. To see this, let $c := (f'_1, \dots, f'_s)$ and $b := (-f'_{s+1} - 1, \dots, -f'_{s'} - 1)$, and note that

$$\begin{aligned} \mathrm{Hom}(\mathcal{L}', \alpha_* \mathcal{L}) &= \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^{1 \times (n-1)}}(-f' - \mathbf{1}), \alpha_* \mathcal{O}_{\mathbb{P}^{1 \times s}}(-\tau(f) - \mathbf{1})) \\ &\cong (\alpha_* \mathcal{O}_{\mathbb{P}^{1 \times s}}(c - \tau(f))) (\mathbf{0}_s, -b). \end{aligned}$$

By Remark 6.10, $\mathrm{Hom}(\mathcal{L}', \alpha_* \mathcal{L}) = H^0(\mathbb{P}^{1 \times s}, \mathcal{O}(c - \tau(f)))$, which is nonzero as $\tau(f) \preceq c$. \blacksquare

7. EQUIVARIANT CONSTRUCTION OF MORPHISMS BETWEEN SUPERNATURAL SHEAVES

Throughout this section, we assume that \mathbb{k} is a field of characteristic 0 and that all root sequences have length $n - 1$. Let V be an n -dimensional \mathbb{k} -vector space, identify \mathbb{P}^{n-1} with $\mathbb{P}(V)$, and let \mathcal{Q} denote the tautological quotient bundle of rank $n - 1$ on $\mathbb{P}(V)$. We have a short exact sequence

$$0 \rightarrow \mathcal{O}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{Q} \rightarrow 0.$$

We will use the fact that $\det \mathcal{Q} \cong \mathcal{O}(1) \otimes \wedge^n V$ is a $\mathbf{GL}(V)$ -equivariant isomorphism. For a weakly decreasing sequence λ of non-negative integers, we let \mathbf{S}_λ denote the corresponding Schur functor. See [Wey03, Chapter 2] for more details (since we are working in characteristic 0, the functors K_λ and L_{λ^t} are isomorphic, where λ^t is the transpose partition of λ , and we call this \mathbf{S}_λ). We extend this definition to weakly decreasing sequences λ with possibly negative entries as follows. Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{n-1}$ and define $\mathbf{S}_\lambda \mathcal{Q} := \mathbf{S}_{\lambda - \lambda_{n-1} \mathbf{1}} \mathcal{Q} \otimes (\det \mathcal{Q})^{\lambda_{n-1}}$.

Proof of Theorem 1.4. The reverse implication has been shown in Proposition 5.3. For the forward implication, we proceed in two steps. First, we construct equivariant supernatural bundles \mathcal{E}' and \mathcal{E} with $\mathrm{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$ using the construction in the proof of [ES09, Theorem 6.2]. Second, we use this fact to construct a new supernatural bundle \mathcal{E}'' of type f' such that $\mathrm{Hom}_{\mathbf{GL}(V)}(\mathcal{E}'', \mathcal{E}) \neq 0$. Thus we will ignore powers of the trivial bundle $\wedge^n V$ that appear in the first step.

Write $N_i = f'_i - f_i$ and let $\lambda \in \mathbb{Z}^{n-1}$ be the partition defined by

$$\lambda_i := f_1 - f_{n-i} - n + 1 + i \quad \text{for } 1 \leq i \leq n - 1.$$

Let λ' be the sequence of weakly decreasing integers defined by $\lambda'_{n-i} := \lambda_{n-i} - N_i$ and set

$$\mathcal{E} := \mathbf{S}_\lambda \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1) \quad \text{and} \quad \mathcal{E}' := \mathbf{S}_{\lambda'} \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1).$$

Observe that $\mathbf{S}_{\lambda'} \mathcal{Q} \otimes \mathcal{O}(-f_1 - 1) \cong \mathbf{S}_{\lambda' + N_1 \mathbf{1}} \mathcal{Q} \otimes \mathcal{O}(-f'_1 - 1)$. Hence by the Borel–Weil–Bott theorem [Wey03, Corollary 4.1.9], \mathcal{E} and \mathcal{E}' are supernatural vector bundles of types f and f' , respectively.

To compute $\text{Hom}(\mathcal{E}', \mathcal{E})$, let $\lambda'' := \lambda' + N_1 \cdot \mathbf{1}$. Define λ^c to be the complement of λ inside of the $(n-1) \times \lambda_1$ rectangle, so $\lambda_j^c = \lambda_1 - \lambda_{n-j}$ for $1 \leq j \leq n-1$. Then $\mathbf{S}_\lambda \mathcal{Q} \cong \mathbf{S}_{\lambda^c} \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1)$ by [Wey03, Exercise 2.18]. We then obtain

$$\text{Hom}(\mathcal{E}', \mathcal{E}) \cong \mathbf{S}_{\lambda'} \mathcal{Q}^* \otimes \mathbf{S}_\lambda \mathcal{Q} \cong \mathbf{S}_{\lambda''} \mathcal{Q}^* \otimes \mathbf{S}_{\lambda^c} \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1 + N_1)$$

and seek to show that this bundle has a nonzero global section.

Fix μ so that $\mathbf{S}_\mu \mathcal{Q}^*$ is a direct summand of $\mathbf{S}_{\lambda''} \mathcal{Q}^* \otimes \mathbf{S}_{\lambda^c} \mathcal{Q}^*$. The Borel–Weil–Bott Theorem [Wey03, Corollary 4.1.9] shows that $\mathbf{S}_\mu \mathcal{Q}^* \otimes \mathcal{O}(\lambda_1 + N_1)$ has nonzero sections if and only if $\lambda_1 + N_1 \geq \mu_1$. This is equivalent to μ being inside of a $(n-1) \times (\lambda_1 + N_1)$ rectangle. By [Ful97, §9.4], the existence of such a μ is equivalent to the condition

$$(7.1) \quad \lambda_i'' + \lambda_{n-i}^c \leq \lambda_1 + N_1 \quad \text{for } i = 1, \dots, n-1.$$

Since $\lambda_i'' + \lambda_{n-i}^c = \lambda_1 + N_1 - N_{n-i}$, we see that (7.1) holds for all i , and thus $\text{Hom}(\mathcal{E}', \mathcal{E}) \neq 0$.

For the second step, replace \mathcal{E}' by $\mathcal{E}'' := \mathcal{E}' \otimes \text{Hom}(\mathcal{E}', \mathcal{E})$, where we view $\text{Hom}(\mathcal{E}', \mathcal{E})$ as a trivial bundle over $\mathbb{P}(V)$. Note that

$$H^i(\mathbb{P}(V), \mathcal{E}''(j)) \cong H^i(\mathbb{P}(V), \mathcal{E}'(j)) \otimes \text{Hom}(\mathcal{E}', \mathcal{E})$$

for all i, j , and hence \mathcal{E}'' is also supernatural of type f' . The space of sections $\text{Hom}(\mathcal{E}'', \mathcal{E})$ is $\text{Hom}(\mathcal{E}', \mathcal{E})^* \otimes \text{Hom}(\mathcal{E}', \mathcal{E})$, which contains the $\mathbf{GL}(V)$ -invariant section corresponding to the evaluation map. This gives a nonzero $\mathbf{GL}(V)$ -equivariant map $\mathcal{E}'' \rightarrow \mathcal{E}$. ■

Example 7.2. We reconsider Example 6.4 in the equivariant context. Here we will not ignore powers of $\bigwedge^n V$. Let $n = 4$ and $f = (-2, -3, -4, -5) \preceq f' = (-1, -2, -3, -4)$. With notation as in the proof of Theorem 1.4, we have $N = (1, 1, 1, 1)$, $\lambda = (0, 0, 0, 0)$, $\lambda' = (-1, -1, -1, -1)$,

$$\mathcal{E} = \mathbf{S}_{(0,0,0,0)} \mathcal{Q} \otimes \mathcal{O}(2-1) = \mathcal{O}(1), \quad \text{and}$$

$$\mathcal{E}' = \mathbf{S}_{(-1,-1,-1,-1)} \mathcal{Q} \otimes \mathcal{O}(2-1) = \left(\mathcal{O}(-1) \otimes \left(\bigwedge^4 V \right)^{-1} \right) \otimes \mathcal{O}(1) = \left(\bigwedge^4 V \right)^{-1} \otimes \mathcal{O}.$$

Since $\lambda^c = (0, 0, 0, 0) = \lambda''$, we see that

$$\text{Hom}(\mathcal{E}', \mathcal{E}) \cong \mathcal{O}(1) \otimes \bigwedge^4 V,$$

which certainly has nonzero global sections. In fact, $\text{Hom}(\mathcal{E}', \mathcal{E}) \cong V \otimes \bigwedge^4 V$. Note, however, that this implies that there is no nonzero equivariant morphism from \mathcal{E}' to \mathcal{E} . We thus set $\mathcal{E}'' := \mathcal{E} \otimes \text{Hom}(\mathcal{E}', \mathcal{E})$. Then $\text{Hom}(\mathcal{E}'', \mathcal{E}) \cong V^* \otimes V$, and our desired nonzero equivariant morphism is given by the trace element. ■

8. REMARKS ON OTHER GRADED RINGS

Given any graded ring R , one could try to use an analog of Theorem 1.1 to induce a partial order on the extremal rays of the cone of Betti diagrams over R . This application has already proven useful in a couple of the other cases where Boij–Söderberg has been studied. In this section, we provide a sketch of some of these applications.

Example 8.1. We first consider an example involving hypersurface rings over $\mathbb{k}[x, y]$. Let $f \in \mathbb{k}[x, y]$ be a quadric polynomial, and set $R := \mathbb{k}[x, y]/\langle f \rangle$. The cone of Betti diagrams

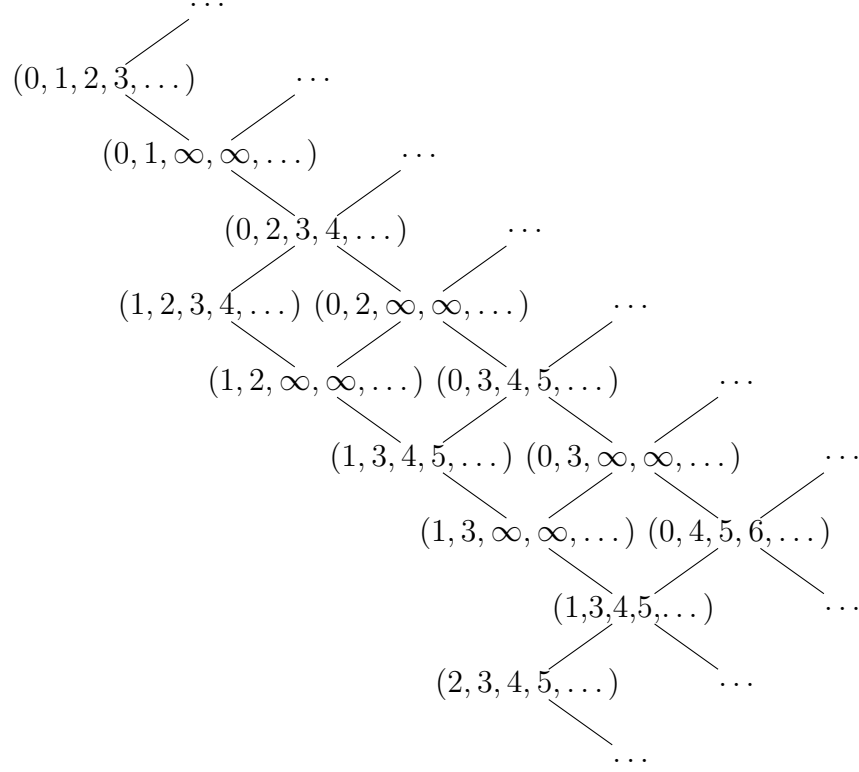


FIGURE 3. For the hypersurface ring R , this partial order provides a simplicial fan structure, as illustrated in [BBEG11] and discussed in Example 8.1. The partial order is determined by an analog of Theorem 1.1.

over R is described in detail in [BBEG11]. The extremal rays still correspond to Cohen–Macaulay modules with pure resolutions, though some of the degrees are infinite in length.

- (i) *Finite pure resolutions.* For example, if h is a degree 7 polynomial that is not divisible by f , then the free resolution of $R/\langle h \rangle$ is

$$R \leftarrow R(-7) \leftarrow 0.$$

Following the notation of Section 2, we denote such a resolution by its corresponding degree sequence, i.e., $(0, 7, \infty, \infty, \dots)$.

- (ii) *Infinite pure resolutions.* For example, the free resolution of the R -module $R/\langle x, y \rangle$ is

$$R \leftarrow R^2(-1) \leftarrow R^2(-2) \leftarrow R^2(-3) \leftarrow \dots.$$

We denote this by its corresponding degree sequence, i.e., $(0, 1, 2, 3, \dots)$.

There are two possible partial orders for these extremal rays:

- $\rho_d \preceq \rho_{d'}$ if $d_i \leq d'_i$ for all i .
- $\rho_d \preceq \rho_{d'}$ if there exist Cohen–Macaulay R -modules M and M' with pure resolutions of types d and d' , respectively, with $\text{Hom}_R(M', M)_{\leq 0} \neq 0$.

In contrast with the case of the polynomial ring, these partial orders are genuinely different. Only the second partial order leads to a greedy algorithm for decomposing Betti diagrams over R , in parallel to [ES09, Decomposition Algorithm]. This also provides an analog of the Multiplicity Conjecture for R . ■

Example 8.2. We now consider $S = \mathbb{k}[x, y]$ with the \mathbb{Z}^2 -grading $\deg(x) := (1, 0)$ and $\deg(y) := (0, 1)$. In general, the cone of bigraded Betti diagrams over S remains poorly understood. However, portions of this cone have been worked out by the first three authors, and we now provide a brief sketch of these unpublished results.

We restrict attention to the cone of Betti diagrams of finite length S -modules M , where all of the Betti numbers of M are concentrated in bidegrees (a, b) with $0 \leq a, b \leq 2$. The extremal rays of this cone may be realized by quotients of monomial ideals of the form m_1/m_2 , where each m_i is a monomial ideal generated by monomials of the form $x^\ell y^k$ with $0 \leq \ell, k \leq 2$. The natural analog of Theorem 1.1 induces a partial order on these rays, which also induces a simplicial structure on this cone of bigraded Betti diagrams. ■

REFERENCES

- [BBEG11] C. Berkesch, J. Burke, D. Erman, and C. Gibbons, *The cone of Betti diagrams over a hypersurface ring of low embedding dimension* (2011). [arXiv:1109.5198v1](#). ↑3, 21
- [BEKS11] C. Berkesch, D. Erman, M. Kummini, and S. V Sam, *Tensor complexes: Multilinear free resolutions constructed from higher tensors* (2011). [arXiv:1101.4604](#). ↑8, 9
- [BS08a] M. Boij and J. Söderberg, *Graded Betti numbers of Cohen–Macaulay modules and the multiplicity conjecture*, J. Lond. Math. Soc. (2) **78** (2008), no. 1, 85–106. ↑4
- [BS08b] ———, *Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen–Macaulay case*, 2008. [arXiv:0803.1645](#). ↑4
- [BH93] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. ↑5
- [EFS03] D. Eisenbud, G. Fløystad, and F.-O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426 (electronic). ↑15
- [EFW11] D. Eisenbud, G. Fløystad, and J. Weyman, *The existence of pure free resolutions*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 905–926. ↑3, 4, 10, 12
- [ES09] D. Eisenbud and F.-O. Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. **22** (2009), no. 3, 859–888. ↑1, 2, 3, 4, 6, 15, 16, 19, 21
- [ES10a] ———, *Cohomology of coherent sheaves and series of supernatural bundles*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 3, 703–722. ↑15
- [ES10b] ———, *Betti numbers of syzygies and cohomology of coherent sheaves*, Proceedings of the International Congress of Mathematicians, 2010. Hyderabad, India. ↑4
- [Ful97] W. Fulton, *Young tableaux, with applications to representation theory and geometry*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. ↑20
- [M2] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*. Available at <http://www.math.uiuc.edu/Macaulay2/>. ↑4
- [SW11] S. V Sam and J. Weyman, *Pieri resolutions for classical groups*, J. Algebra **329** (2011), 222–259. ↑3, 10, 11, 12
- [Wey03] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics, vol. 149, Cambridge University Press, Cambridge, 2003. ↑19, 20

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